Announcements

1) Midterm Up – on CTools under "Assignments" due March 8^{th} (Friday)

2) Office hours change for week after break

No 9:30 office hours, instead 1-2:30 M
4-4:30 W

Note: Abel's Theorem

Extends to intervals of the form $[-R,0]$, $R>0$. We will assume this.

Corollary: If $\sum_{n=0}^{\infty} a_n x^n$ converges on an interval I, then the convergence is uniform on compact subsets of I.

proof: Let K be a compact subset of I.

Case 1: $K = I$. Then $K = [a, b]$ and either $a < 0$ or $b > 0$.

In either case (say $b > 0$ and $b > |a|$), use Abel's theorem on $[0, b]$ to achieve uniform convergence there.

If $0 \le a < b$, then the result follows immediately. If $a < 0$, apply Abel's theorem to $[a, 0]$, then combine the results to obtain uniform convergence on $[a, b] = K = I$.

Case 2: $K \neq I$.

Then K does not contain at least one endpoint, say a $(a<b)$ of I. Let $\alpha = \inf(K)$, $\beta = \sup(K)$. Again suppose $\beta > 0$, $\beta > |\alpha|$.

Since K is compact,
at least one of
α or β (say β)
is contained in I.

Then $\sum_{n=0}^{\infty} a_n x^n$ converges

at β. Therefore,

$\sum_{n=0}^{\infty} a_n x^n$ converges for
all x, $|x| < \beta$

Apply Abel's Theorem

to $[0, \beta]$ to

obtain uniform convergence

of $\displaystyle\sum_{n=0}^{\infty} a_n x^n$.

Subcase 1: $-\beta < \alpha$

Apply Abel's theorem to

$[x, 0]$ where

$-\beta < x < \alpha$.

We then have uniform convergence on $[\alpha, \beta] \supseteq K$

$\implies$ uniform convergence on K.

<u>Subcase 2</u>: $-\beta \geq \alpha$.

If α is contained in I, then

by Abel's theorem again,

the series converges

uniformly on $[\alpha, 0]$

$\Rightarrow$ uniform convergence

on $[\alpha, \beta] \Rightarrow$ uniform

convergence on $K \subseteq [\alpha, \beta]$.

If α is not

contained in I, then

$\alpha \notin K$, contradiction.

Therefore,

$$\sum_{n=0}^{\infty} a_n x^n \text{ converges}$$

uniformly on K. $\square$

Theorem: Suppose

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \text{ on}$$

some interval I.

Then f is continuous on I and differentiable on any interval $(-R, R) \subseteq I$, with

$$\boxed{f'(x) = \sum_{n=1}^{\infty} n a_n x^n.}$$

Step 1: f is continuous on I.

We already know the convergence of $\sum_{n=0}^{\infty} a_n x^n$ is uniform on compact subsets of I by previous corollary.

Let $a<b$ be the endpoints of I.

Then $\sum_{n=0}^{\infty} a_n x^n$

converges uniformly on

$[a+\varepsilon, b-\varepsilon]$ $\forall$

$0 < \varepsilon < \frac{b-a}{2}$. Therefore

f is continuous on

$[a+\varepsilon, b-\varepsilon]$ $\forall$ $0 < \varepsilon < \frac{b-a}{2}$

$\Rightarrow$ f is continuous on

(a, b).

If neither a nor b is in I, then done.

If, say, $a \in I$, then by some form of Abel's Theorem, the series converges uniformly on a closed interval containing a

$\Rightarrow f$ is continuous at a.

<u>Step 2</u>: $\sum_{n=1}^{\infty} n a_n x^{n-1}$

converges on $(-R, R) \subseteq I$.

Write, for $x \in (-R, R)$,

$$|n a_n x^{n-1}|$$
$$= n |a_n| \, |x|^{n-1}$$
$$= n |a_n| \, \left|\frac{x}{t}\right|^{n-1} t^{n-1}$$

for some t, $|x| < t < R$.

Claim: $\Big(\underbrace{n\left|\frac{x}{t}\right|^{n-1}}_{b_n}\Big)_{n=1}^{\infty}$

is bounded.

Why?

Apply the ratio test to

$$\sum_{n=1}^{\infty} n\left|\frac{x}{t}\right|^{n-1}.$$

We get

$$\frac{b_{n+1}}{b_n} = \frac{n+1}{n} \left| \frac{x}{t} \right|$$

$$\longrightarrow \left| \frac{x}{t} \right| = \frac{|x|}{t} < 1$$

$$\Rightarrow \sum_{n=1}^{\infty} b_n \text{ converges}$$

by the ratio test

$$\Rightarrow \lim_{n \to \infty} b_n = 0$$

$$\Rightarrow (b_n)_{n=1}^{\infty} \text{ is bounded.}$$

Then $\exists\ M>0$,

$$b_n \le M \quad \forall\, n \in \mathbb{N}.$$

We apply this to

$$|a_n| \overbrace{n \left|\frac{x}{t}\right|^{n-1}}^{=b_n} t^{n-1}$$

$$\le M\, |a_n|\, t^{n-1}$$

$$= \frac{M}{t}\, |a_n|\, t^n$$

which converges since

$$\sum_{n=0}^{\infty} a_n x^n \text{ converges}$$

absolutely on $(-R, R)$.

Therefore by comparison,

$$\sum_{n=1}^{\infty} a_n \cdot n \cdot x^{n-1}$$

converges on $(-R, R)$.

Step 3: $f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}$

$\forall x \in (-R, R)$.

We know f converges

on some point in I

(actually, on all of I).

We know that if

$P_m(x) = \sum_{n=1}^{m} n a_n x^{n-1}$, P_m

converges uniformly on $(-R, R)$.

Then by our differentiation theorem, the sequence

$$\left(Q_m(x)\right)_{m=0}^{\infty} \text{ with}$$

$$Q_m(x) = \sum_{n=0}^{m} a_n x^n$$

will satisfy $Q_m' = P_m \to \sum_{n=1}^{\infty} n a_n x^{n-1}$

uniformly on $(-R, R)$ and

Q_m converges on $(-R, R)$ to

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \implies$$

f is differentiable on $(-R, R)$ and

$$f'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1}. \quad \square$$